Free Wave in Vacuum

Disipative waves in a charge free media

$$\sigma \neq 0 \qquad \rho_s = 0$$

$$\vec{J}_e = \sigma \vec{E}$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \frac{-\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$= -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

$$= -\mu \sigma \frac{\partial}{\partial t} \vec{E} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} \qquad (\rho_s = 0)$$

$$-\vec{\nabla}^2 \vec{E} = -\mu \sigma \frac{\partial}{\partial t} \vec{E} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$= -\mu \epsilon \left(\frac{\sigma}{\epsilon} \frac{\partial \vec{E}}{\partial t} \right) - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Let $\vec{E} = \mathrm{Re}\left[\vec{E}_s e^{-i\omega t} \right]$

$$E_x(x, y, z, t) = \mathrm{Re}\left[E_{sx} e^{-j\omega t} \right]$$

What is E_s?

E_s is this vector which, when multiplied by $e^{-i\omega t}$ and "take $\mathrm{Re}[\vec{E}_s e^{-i\omega t}]$" you get $\vec{E}$; E_s is a function of x_P, y_P, z_P only not of t

$$\frac{\partial}{\partial t}\vec{E} = \frac{\partial}{\partial t}\mathrm{Re}[\vec{E}_s e^{-i\omega t}]$$

$$= \mathrm{Re}\left[\frac{\partial}{\partial t} E_s e^{-i\omega t}\right]$$

$$= \mathrm{Re}\left[-\omega t E_s e^{-i\omega t}\right]$$

Maxwell in Phasor form

(going to phasors: $\frac{\partial}{\partial t} \rightarrow -j\omega$)

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{\nabla}\cdot\vec{D}_s = \rho_s$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{E}_s = -j\omega\mu\vec{H}_s$$

$$\vec{\nabla}\cdot\vec{H} = 0 \qquad \text{unchanged}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_c + \frac{\partial\vec{D}}{\partial t} \qquad \vec{\nabla}\times\vec{H}_s = \vec{J}_{cs} + j\omega\varepsilon\vec{E}_s$$

Waves in charge free media in phasor form

$$\rho = 0$$

$$\vec{\nabla}\times\vec{E}_s = -j\omega\mu\vec{H}_s$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}_s) = -j\omega\mu(\nabla\times\vec{H}_s)$$

$$= -j\omega\mu(\vec{J}_{cs} + j\omega\varepsilon\vec{E}_s)$$

$$-\vec{\nabla}^2 \vec{E}_s = -j\omega\mu(\sigma \vec{E}_s + j\omega\varepsilon\vec{E}_s)$$
$$= (-j\omega\mu\sigma + \omega^2\mu\varepsilon)\vec{E}_s$$
$$= \omega^2\mu\left(\varepsilon - \frac{j\sigma}{\omega}\right)\vec{E}_s$$

let $\varepsilon_c = \varepsilon - \frac{j\sigma}{\omega} = \varepsilon\left(1 - \frac{j\sigma}{\varepsilon\omega}\right)$

$$-\vec{\nabla}^2 \vec{E}_s = \omega^2\mu\varepsilon_c\vec{E}_s$$

Note: if σ_0 then $\varepsilon_c = \varepsilon$ and $\omega^2\mu\varepsilon$ is real

$$\vec{\nabla}^2 \vec{E}_s + \omega^2\mu\varepsilon_c\vec{E}_s = 0$$

if $\sigma = 0$ then $\varepsilon_c = \varepsilon$
and $\omega^2\mu\varepsilon$ can be solved
for ω^2

Introduce the complex parameter γ

$$\gamma^2 = -\omega^2\mu\varepsilon_c = -\omega^2\mu\left(\varepsilon - \frac{j\sigma}{\omega}\right)$$

$$\vec{\nabla}^2 \vec{E}_s - \gamma^2\vec{E}_s = 0$$

Solutions are therefore

$$\sim e^{\pm\gamma(\hat{n}\cdot\vec{r})}$$

$\hat{n}\cdot\vec{r} = n_x x + n_y y + n_z z$ & $n_x^2 + n_y^2 + n_z^2 = 1$

$$\frac{\partial^2}{\partial x^2}\left[E_o e^{\pm\gamma(\hat{n}\cdot\vec{r})}\right] + \frac{\partial^2}{\partial y^2}\left[E_o e^{\pm\gamma(\hat{n}\cdot\vec{r})}\right]$$
$$+ \frac{\partial^2}{\partial z^2}\left[E_o e^{\pm\gamma(\hat{n}\cdot\vec{r})}\right] = \gamma^2 E_o e^{\pm\gamma(\hat{n}\cdot\vec{r})}$$

The type of solution $\sim e^{\pm\gamma(\hat{n}\cdot\vec{r})}$ is called the plane wave because, on a fixed plane determined by $\hat{n}\cdot\vec{r} = \text{constant}$,

$\vec{E}_s$, $\vec{H}_s$ have constant amplitudes.

For the moment, suppose $\hat{n}\cdot\vec{r} = z$

Some properties:

$\gamma = \alpha + j\beta$ $\qquad\qquad$ $\vec{E}_o$: constant

The physical feild is

$$\vec{E}(z,t) = \mathrm{Re}\left[\vec{E}_s e^{+j\omega t}\right]$$
$$= \mathrm{Re}\left[\vec{E}_o e^{\alpha z} e^{j(Bz+\omega t)}\right]$$

For simplicity, Assume $\vec{E}_o$ is a real vector

$$= \vec{E}_o e^{\alpha z}\cos(Bz+\omega t)$$

Beware: $\vec{E}_s = \mathrm{Re}\left[\vec{E}_s e^{+j\omega t}\right]$; this change the wave, it now propogates to the left instead of the right, This in essence changes to polarity of α.

$t_0 = 0 \qquad z = -5$

$t_1 = 1 \qquad z = -7$

$e^{\alpha z}$ becomes smaller as z becomes smaller

$e^{-\gamma z}$

$$\vec{E}(z,t) = Re\left[\vec{E}_o e^{-\alpha z} e^{j(-\beta z + \omega t)}\right]$$
$$= \vec{E}_o e^{-\alpha z} \cos(\omega t - \beta z)$$